

Solution of Homological Algebra Week 2

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1 Some properties of abelian category

Recall that in an abelian category (1) every map has kernel and cokernel, (2) monic is kernel of its cokernel, epic is cokernel of its kernel.

(a) To prove f is monic when $0 \rightarrow A$ is a kernel of f , take any object C , and take $g_1, g_2 : C \rightarrow A$ two morphisms that satisfy $f \circ g_1 = f \circ g_2$. By abelian group structure of Hom and distribution law, $0 = f \circ g_1 - f \circ g_2 = f \circ (g_1 - g_2)$. By the universal property of kernel, there exists a unique k that satisfy $g_1 - g_2 = h \circ k = 0$, because $h = 0$. So $g_1 - g_2 = 0$, $g_1 = g_2$ which proves that f is monic.

$$\begin{array}{ccccc}
 & & 0 = \ker(f) & & \\
 & \nearrow k & \downarrow h & & \\
 C & \xrightarrow{g_1 - g_2} & A & \xrightarrow{f} & B
 \end{array}$$

To prove that f is epi, when $B \rightarrow 0$ is a cokernel of f . Similarly take any object C , suppose $g_1, g_2 : B \rightarrow C$ two morphisms that satisfy $g_1 \circ f = g_2 \circ f$, then the universal property of cokernel shows that there exists a unique k such that $g_1 - g_2 = k \circ h = 0$. So $g_1 = g_2$ which proves that f is epi.

$$\begin{array}{ccccc}
 & & 0 = \text{coker}(f) & & \\
 & \nwarrow k & \uparrow h & & \\
 C & \xleftarrow{g_1 - g_2} & B & \xleftarrow{f} & A
 \end{array}$$

(b) To prove isomorphism, it suffices to construct $g : B \rightarrow A$ such that $g \circ f = \text{id}_A$. First take coker, im of f by property (1).

$$\begin{array}{ccccc}
 & & \text{im}(f) = \ker(\text{coker}(f)) & & \\
 & & \downarrow & & \\
 A & \xrightarrow{f} & B & \longrightarrow & \text{coker}(f)
 \end{array}$$

Then since f is monic, it is kernel of $B \rightarrow \text{coker}(f)$, $\text{im}(f)$ is also kernel of $B \rightarrow \text{coker}(f)$, by universal property of kernel, they are isomorphism, and the diagram commutes.

$$\begin{array}{ccccc}
& & im(f) = ker(coker(f)) & & \\
& \nearrow \sim & \downarrow & & \\
A & \xrightarrow{f} & B & \longrightarrow & coker(f)
\end{array}$$

Finally, since f is epi and

$$A \rightarrow B \rightarrow coker(f) = A \leftrightarrow im(f) \rightarrow B \rightarrow coker(f) = 0$$

$$B \rightarrow coker(f) = 0$$

By adding $id_B : B \rightarrow B$, and universal property of kernel, there exists a unique h such that the diagram commutes.

$$\begin{array}{ccccc}
& & im(f) = ker(coker(f)) & & \\
& \nearrow i \sim & \downarrow & & \\
A & \xrightarrow{f} & B & \xrightarrow{id} & B \longrightarrow coker(f)
\end{array}$$

Take $g = i \circ h$, then $g \circ f = id_A$.

(c) The induced morphism is given by the universal property of kernel.

$$\begin{array}{ccccc}
& & im(f) & & \\
& \nearrow g & \downarrow & & \\
A & \xrightarrow{f} & B & \longrightarrow & coker(f)
\end{array}$$

To prove epi, take any object C and $h_1, h_2 : im(f) \rightarrow C$ such that $h_1 \circ g = h_2 \circ g$, consider the difference $h_1 - h_2$.

$$\begin{array}{ccccc}
& & im(f) & \xrightarrow{h_1 - h_2} & C \\
& \nearrow g & \downarrow & & \\
A & \xrightarrow{f} & B & \longrightarrow & coker(f)
\end{array}$$

Then take the kernel of $h_1 - h_2$, since $(h_1 - h_2) \circ g = 0$, by the universal property of kernel, there exists a unique $A \rightarrow ker(h_1 - h_2)$ such that the diagram commutes.

$$\begin{array}{ccccc}
ker(h_1 - h_2) & \xrightarrow{i} & im(f) & \xrightarrow{h_1 - h_2} & C \\
\uparrow \text{---} & \nearrow g & \downarrow j & & \\
A & \xrightarrow{f} & B & \longrightarrow & coker(f)
\end{array}$$

Since i, j are both kernels, they are monics (two morphisms $a_1, a_2 : C \rightarrow ker(f)$, then the uniqueness of universal property shows that they are the same), and $j \circ i$ is monic (easy to check that the composition of monics is monic by using the definition of monic twice). By property (2), $j \circ i$ is kernel of its cokernel. And because $A \rightarrow B \rightarrow coker(j \circ i) = 0$, by the universal property of cokernel, there exists a unique $coker(f) \rightarrow coker(j \circ i)$ such that the diagram commutes.

$$\begin{array}{ccccc}
\ker(h_1 - h_2) & \xrightarrow{i} & \operatorname{im}(f) & \xrightarrow{h_1 - h_2} & C \\
\uparrow & \nearrow j \circ i & \downarrow j & & \\
A & \xrightarrow{f} & B & \longrightarrow & \operatorname{coker}(f) \\
& & & \searrow & \downarrow \text{dashed} \\
& & & & \operatorname{coker}(j \circ i)
\end{array}$$

Then $\operatorname{im}(f) \rightarrow B \rightarrow \operatorname{coker}(j \circ i) = 0$, since $j \circ i$ is a kernel, there exists a unique i' such that the diagram commutes, that is to say i is an isomorphism.

$$\begin{array}{ccccc}
\ker(h_1 - h_2) & \xleftarrow{i} & \operatorname{im}(f) & \xrightarrow{h_1 - h_2} & C \\
\uparrow & \nearrow i' & \downarrow j & & \\
A & \xrightarrow{f} & B & \longrightarrow & \operatorname{coker}(f) \\
& & & \searrow & \downarrow \text{dashed} \\
& & & & \operatorname{coker}(j \circ i)
\end{array}$$

Finally, take $\operatorname{id}_{\operatorname{im}(f)}$, $(h_1 - h_2) = (h_1 - h_2) \circ \operatorname{id}_{\operatorname{im}(f)} = 0$, this proves that g is epi.

2 Functor Category of Abelian Category is Abelian

First check that $\mathcal{A}^{\mathcal{I}}$ is additive. For every set $\operatorname{Hom}_{\mathcal{A}^{\mathcal{I}}}(C, D)$, its structure of abelian group is given by taking I an object in $\mathcal{I}$, then $(\eta_I + \phi_I)(c) := \eta_I(c) + \phi_I(c)$, and the composition $(\Psi_I \circ \eta_I)(c) = \Psi_I(\eta_I(c))$, then the abelian group structure and distribution law hold by the corresponding properties of $\mathcal{A}$. The zero object is $Z : \mathcal{I} \rightarrow \mathcal{A}$, for any I an object of $\mathcal{I}$, $Z(I) = 0_{\mathcal{A}}$. The product is $F \times G(I) = F(I) \times_{\mathcal{A}} G(I)$.

Then check that $\mathcal{A}^{\mathcal{I}}$ is abelian. For any map $\eta \in \operatorname{Hom}_{\mathcal{A}^{\mathcal{I}}}(C, D)$, take the kernel $\epsilon \in \operatorname{Hom}_{\mathcal{A}^{\mathcal{I}}}(E, C)$ and cokernel $\iota \in \operatorname{Hom}_{\mathcal{A}^{\mathcal{I}}}(D, F)$ for any I an object of $\mathcal{I}$ to be : $E(I) := \ker(\eta_I : C(I) \rightarrow D(I))$, $\epsilon_I : E(I) \rightarrow C(I)$ the corresponding kernel morphism, and $F(I) := \operatorname{coker}(\eta_I : C(I) \rightarrow D(I))$, $\iota_I : D(I) \rightarrow F(I)$ the corresponding cokernel morphism. They satisfy the universal properties of kernel and cokernel just by the universal properties in $\mathcal{A}$, and similarly this gives the kernel and cokernel of every map. Monic (or epi) η in $\mathcal{A}^{\mathcal{I}}$ is just to say that for any I an object in $\mathcal{I}$, η_I is monic (or epi), so by the corresponding condition of abelian category of $\mathcal{A}$, each η_I is kernel of its cokernel (or cokernel of its kernel), so η satisfies the same condition. So $\mathcal{A}^{\mathcal{I}}$ is an abelian category.

3 Exactness of Adjoint Functors

To prove that R is left exact, take exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C$$

For any M an object in $\mathcal{A}$, act the exact sequence by functor $\text{Hom}_{\mathcal{B}}(L(M), \cdot)$ which is left exact, this gives an exact sequence

$$0 \rightarrow \text{Hom}_{\mathcal{B}}(L(M), A) \rightarrow \text{Hom}_{\mathcal{B}}(L(M), B) \rightarrow \text{Hom}_{\mathcal{B}}(L(M), C)$$

Then act by the natural isomorphism τ , this gives an exact sequence

$$0 \rightarrow \text{Hom}_{\mathcal{A}}(M, R(A)) \rightarrow \text{Hom}_{\mathcal{A}}(M, R(B)) \rightarrow \text{Hom}_{\mathcal{A}}(M, R(C))$$

By Yoneda lemma, this shows that $0 \rightarrow R(A) \rightarrow R(B) \rightarrow R(C)$ is exact. So R is left exact.

To prove that L is right exact, just to take everything in the opposite category. Take exact sequence

$$0 \rightarrow C^{op} \rightarrow B^{op} \rightarrow A^{op}$$

For any M an object in $\mathcal{B}^{op}$, act the exact sequence by functor $\text{Hom}_{\mathcal{A}^{op}}(R^{op}M, \cdot)$ which is left exact, this gives an exact sequence

$$0 \rightarrow \text{Hom}_{\mathcal{A}^{op}}(R^{op}M, C^{op}) \rightarrow \text{Hom}_{\mathcal{A}^{op}}(R^{op}M, B^{op}) \rightarrow \text{Hom}_{\mathcal{A}^{op}}(R^{op}M, A^{op})$$

Then act by the natural isomorphism τ^{op} , this gives an exact sequence

$$0 \rightarrow \text{Hom}_{\mathcal{B}^{op}}(M, (LC)^{op}) \rightarrow \text{Hom}_{\mathcal{B}^{op}}(M, (LB)^{op}) \rightarrow \text{Hom}_{\mathcal{B}^{op}}(M, (LA)^{op})$$

By Yoneda lemma, this shows that $0 \rightarrow (LC)^{op} \rightarrow (LB)^{op} \rightarrow (LA)^{op}$ is exact. Take the opposite category again, $LA \rightarrow LB \rightarrow LC \rightarrow 0$ is exact. So L is right exact.

4 Five Lemma

To prove this result, it is easier to embed the abelian category in some R -module category to apply diagram-chasing. This embedding can be realized by Freyd-Mitchell embedding theorem, as long as the category is a small category.

To reduce to a small category, we try to construct a small abelian category containing all the objects in the diagram. Take $\mathcal{C}_0$ the full subcategory of $\mathcal{A}$ which contains objects $S_0 = \{0, A, B, C, D, E, A', B', C', D', E'\}$. Then construct $\mathcal{C}_{n+1}$ as full subcategory of $\mathcal{A}$ which contains objects

$$S_{n+1} = S_n \cup \{ker(f) | f \in \text{Hom}_{\mathcal{A}}(J, K), J, K \in S_n\} \cup \{coker(f) |$$

$$f \in \text{Hom}_{\mathcal{A}}(J, K), J, K \in S_n\} \cup \{J \times K \mid J, K \in S_n\}$$

Finally get the limit $\mathcal{C}_\infty$ which contains objects $S_\infty = \bigcup_{\mathbb{N}} S_n$, and since S_n are all sets, S_∞ is also a set, so $\mathcal{C}_\infty$ is a small category. And because $\mathcal{C}_\infty$ is full subcategory of $\mathcal{A}$, it is $\mathbb{A}\mathbf{b}$ -category, and by the construction, it satisfies all the conditions of abelian category.

So without loss of generality, assume $\mathcal{A}$ is small, so by Freyd-Mitchell embedding theorem, there exists some ring R , and a fully faithful exact embedding functor $i : \mathcal{A} \rightarrow \mathbf{R} - \text{mod}$. By this embedding, the monic ($\ker = 0$), epi ($\text{coker} = 0$) and exact property is just the same to check in a full subcategory of $\mathbf{R} - \text{mod}$, where monic (or epi) equals injective (or surjective) morphism. For any morphism $f : A \rightarrow B$, take its kernel and cokernel, this gives an exact sequence

$$0 \rightarrow \ker(f) \rightarrow A \rightarrow B \rightarrow \text{coker}(f) \rightarrow 0$$

and then apply i , which is exact, this gives the following exact sequence

$$0 \rightarrow i(\ker(f)) \rightarrow i(A) \rightarrow i(B) \rightarrow i(\text{coker}(f)) \rightarrow 0$$

This shows that we can check kernel, cokernel and exactness in the full subcategory of $\mathbf{R} - \text{mod}$.

Check that c is monic in $\mathbf{R} - \text{mod}$. $\forall x \in C', c(x) = 0$, take the image y of x in D' , then $d(y) = 0$. Since d is monic, $y = 0$.

$$\begin{array}{ccc} x \in C' & \longrightarrow & y = 0 \\ \downarrow c & & \downarrow \\ 0 & \longrightarrow & 0 \end{array}$$

By the exactness of the first row, there exists $w \in B'$ such that x is the image of w . Take $\bar{w} = b(w)$, then its image in C is zero, so by exactness of second row, there exists $\bar{v} \in A$ such that $\bar{w}$ is the image of $\bar{v}$.

$$\begin{array}{ccccccc} w & \longrightarrow & x \in C' & \longrightarrow & y = 0 \\ \downarrow & & \downarrow c & & \downarrow \\ \bar{v} & \longrightarrow & \bar{w} & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

Since a is epi, the preimage $v \in A'$ of $\bar{v}$ exists, and the image of v in B equals $\bar{w}$, and since b is monic, the image of v in B' is w .

$$\begin{array}{ccccccc} v & \longrightarrow & w & \longrightarrow & x \in C' & \longrightarrow & y = 0 \\ \downarrow & & \downarrow & & \downarrow c & & \downarrow \\ \bar{v} & \longrightarrow & \bar{w} & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

So x is the image of v , which is zero. This proves that c is injective.

Check c is epi in $\mathbf{R} - \text{mod}$. Fix an element $x \in C$, it suffices to show that there is a preimage of x in C' . Take the images of x in D, E , since d is epi, there exists $y' \in D'$ such that $d(y') = y$, and since e is monic, the image of y' in E' is zero, so there exists $x' \in C'$ such that its image in D' is y' .

$$\begin{array}{ccccc}
x' & \longrightarrow & y' & \longrightarrow & 0 \\
& & \downarrow & & \downarrow \\
x & \longrightarrow & y & \longrightarrow & 0
\end{array}$$

Take $u = x - c(x') \in C$, its image in D is zero by construction, so there exists $w \in B$ such that its image in C is u . Since b is epic, there exists $w' \in B'$ such that $b(w') = w$, take its image u' in C' .

$$\begin{array}{ccccc}
w' & \longrightarrow & u' & & \\
\downarrow & & & & \\
w & \longrightarrow & u = x - c(x') & \longrightarrow & 0
\end{array}$$

Finally, the preimage of x in C' is just $u' + x'$. This proves that c is surjective.

When a, b, d, e are isomorphisms, by the previous proof, c is monic and epi, so c is an isomorphism.